END SEMESTER EXAMINATION - COMMUTATIVE ALGEBRA - MMATH - 29 NOVEMBER 2010

Attempt all questions. All rings considered are commutative with 1. Total Marks - 50. Time - 3 hrs.

(1) Let $A = \{f(z) \in \mathbb{C}[z]$ such that $\frac{df}{dz}(0) = \frac{d^2f}{dz^2}(0) = \ldots = \frac{d^kf}{dz^k}(0) = 0\}$ where $k \geq 1$ is an integer. Is A a Noetherian ring? Justify your answer. (5 marks)

(2) Let M be a Noetherian A-module and let $Ann(M)$ be the ideal $\{a \in A | am = 0$ for all $m \in M\}$. Prove that $A/Ann(M)$ is a Noetherian ring. If we replace Noetherian by Artinian in this result, is it still true? (5+5 marks)

(3) Let A be a Noetherian ring. Prove that the following are equivalent
  (a) A is Artinian,
  (b) $Spec(A)$ is discrete and finite,
  (c) $Spec(A)$ is discrete.

Give an example of a Noetherian ring A such that $Spec(A)$ is finite but not discrete. Give an example of a non-Noetherian ring A with only one prime ideal (in particular $Spec(A)$ is finite and discrete). (6+2+2 marks)

(4) Let A be a ring and let P be a prime ideal of A. Let $P^{(n)}$ be the ideal $(P^nA_P)^c$, that is, $P^{(n)}$ is the contraction of the extension of the ideal P^n with respect to the natural homomorphism $A \to A_P$. Show that $P^{(n)}$ is a P-primary ideal. Show also that, if P^n has a primary decomposition then $P^{(n)}$ is it's P-primary component. (5+5 marks)

(5) Let A be a subring of a ring B such that B is integral over A. Prove that the induced map $Spec(B) \to Spec(A)$ induces a map $Max(B) \to Max(A)$. Prove that the map $Spec(B) \to Spec(A)$ is surjective. Prove that the map $Spec(B) \to Spec(A)$ is a closed mapping with respect to the Zariski topology on both sides. (5+5+5 marks)